- Permanyer, I. (2017) 'Benchmark inequality measures'. Presented in Session 3.1. 7th Meeting of the Society for the Study of Economic Inequality (ECINEQ). Society for the Study of Economic Inequality. New York (USA), July 17-19. http://ecineq.org/ecineq_nyc17/documents/ECINEQ%202017%20NYC%20booklet.pdf

Benchmark inequality measures

Iñaki Permanyer

Centre d'Estudis Demogràfics

Edifici E-2, Campus de la UAB

E-mail: inaki.permanyer@uab.es

Phone: +34 935813060

Abstract: In measuring the variability of a bounded variable such as health status, the use of traditional income inequality measures poses several problems. On the one hand, the bounded nature of the variable generates a strong relationship between its mean and the corresponding inequality levels (i.e. when the mean approaches either the lower or the upper bounds, inequality mechanically goes to zero). On the other hand, the lack of consistency between achievement and shortfall distributions precludes the use of classical measures of relative inequality. In this paper we propose new inequality indices that aim at capturing the intuitions of relative inequality but adapted to the context of bounded variables (the so-called benchmark inequality measures). The benchmark inequality indices proposed here simultaneously solve both problems: they consistently rank achievement and shortfall distributions and take into account the fact that when the mean of a bounded distribution approaches some of its bounds, the scope for potential variability is substantially reduced.

Keywords: Inequality measurement, Bounded variables, Attainment, Shortfall, Consistency.

JEL Codes: D63, I31

1. Introduction

The analysis of non-pecuniary dimensions of well-being often forces researchers to work with bounded variables, like health status, nutritional intake, educational attainment, and so on. In this paper we contend that the traditional tools of (income) inequality analysis can be problematic when exploring the variability of bounded variables because two problems of different nature arise: (i) the 'boundary effects' and (ii) the 'consistency' problems. By 'boundary effects' we refer to the clustering that takes place across observations when the mean of the distribution converges towards some of its bounds. When this happens, the corresponding inequality levels mechanically go to zero – simply because there is no room for further variation – an issue that complicates comparisons of the levels of inequality for distributions with different means. Therefore, one is left pondering whether the convergence findings commonly reported for many education or health variables (e.g. Neumayer 2003, Sutcliffe 2004, Kenny 2005, Dorius 2013) are purely mechanically driven by the fact that the mean might be approaching some of its bounds. In these circumstances, it is not clear that studying the distribution of a bounded variable can provide new insights above what we already know from studying the values of mean alone. One of the main goals of this paper is to make room for the possibility of factoring out the influence of the mean when computing the inequality levels of bounded variables.

In the classical unbounded framework (e.g. in the case of income inequality measurement), one typically factors out the effects of differing means by working with relative measures. Unfortunately, this approach can be problematic when the variable we are working with is bounded. In that context, it is *a priori* possible to focus on the distribution of achievements or on the corresponding distribution of shortfalls with respect to the upper bound[1] . As highlighted by Clarke et al (2002), Erreygers (2009), Lambert and Zheng (2011) among many others, relative inequality measures fail to consistently rank distributions depending on whether the latter are measured as attainments of a given indicator or as shortfalls with respect to its upper bound. This is the so-called '(in)consistency problem', which, far from being a mere academic curiosity, poses several practical challenges to the study of inequality of bounded variables. Roughly speaking, the main conclusion of this strand of research is that in order to satisfy certain consistency conditions it is necessary to work with absolute measures of inequality. Traditional relative measures of inequality fail to be consistent because the relativity depends on the end from which one considers the distributions.

The lack of relative measures of inequality for bounded variables is unfortunate because we miss the kind of intuitions provided by those measures that are so useful to compare distributions with differing means. The main contribution of this paper is to complement the aforementioned absolute measures of inequality with another class of inequality measures that takes into consideration the fact the range of variation of a bounded variable is conditioned by the value of the mean. Our approach is simple: we compare observed inequality levels with the maximal inequality levels that could be possibly observed with the same index in another hypothetical distribution having the same mean (the so-called 'benchmark distribution') – an approach that is only feasible in the bounded case. This way we generate new families of inequality indices that are always bounded between 0 and 1 that facilitate the comparison of distributions with alternative means – so they can be naturally thought of as the 'relative version' of inequality indices for the bounded case. It turns out that the

[1] To illustrate: improvements in the coverage of public health plans could be assessed via the percentage of vaccinated children (an achievement indicator) or through the percentage of unvaccinated children (a shortfall indicator).

new indices satisfy the basic requirements of inequality measurement (i.e. they are proper inequality measures) and are neither affected by the boundary effects nor the (in)consistency problems.

After defining our new benchmark inequality measures we illustrate how they perform empirically. For that purpose we study the distribution of under five mortality rates around the world from 1950 to 2010 applying our new inequality measures and comparing them with respect to currently existing approaches. These results will be used to revisit the debate on whether the world is converging or not on several health indicators – an issue that has attracted a good deal of attention during the last years to gauge the welfare impacts of the economic globalization process (see, among many others, Easterlin 2000, Neumayer 2003, Sutcliffe 2004, Kenny 2005, Clark 2011). The rest of the paper is organized as follows. In section 2 we introduce the notation and basic definitions. In Section 3 we present our new benchmark inequality indices. Section 4 shows the empirical illustration and Section 5 concludes. The proofs are relegated to the appendix.

2. Notation and basic definitions

In this paper the different units of analysis $\{1, \ldots, n\}$ ($n \in \mathbb{N}$) will be referred to as 'countries', even if in practice one might actually work with any other groups, such as municipalities, households or individuals. An *achievement distribution* across our n units of analysis is represented by a vector $\mathbf{x} = (x_1, \ldots, x_n) \in D^n$, with $D^n = \mathbb{R}^n_+$, where x_i represents country i's achievement. The set of all possible distributions is $D = \bigcup_{n \geq 2} D^n$. Given any $\mathbf{x} \in D$, let $\mu(\mathbf{x})$ represent the mean of the distribution. For any $k \in \mathbb{R}$, $\mathbf{x} + k$ denotes the vector where each component is $x_i + k$. Analogously, for any $\lambda > 0$, $\lambda\mathbf{x}$ is the vector where each component equals λx_i. For any $c \in \mathbb{R}$, $\mathbf{c} := (c, \ldots, c)$ is the vector where each component equals c.

The achievement level of a given country will be measured by means of an indicator that in this paper we assume to be bounded from above and below. We assume that the lower bound is set to 0. We will denote by U the upper bound ($U > 0$). We also assume that *both* the lower and upper bounds can be attained by the underlying indicator. Given any $U > 0$, let D^U represent the set of distributions for which U is an upper bound, that is: $D^U := \{\mathbf{x} \in D | 0 \leq x_i \leq U \ \forall i\}$. The *shortfall distribution* associated with $\mathbf{x} \in D^U$ will be denoted as $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n_+$ with $s_i = U - x_i$ representing country i's shortfall. Given two distributions $\mathbf{x}, \mathbf{x}' \in D$, we say that $\mathbf{x}'$ is obtained from $\mathbf{x}$ by a *progressive transfer* if there are two countries $i, j \in \{1, \ldots, n\}$ and $k > 0$ such that $x'_i = x_i + k \leq x_j - k = x'_j$ and $x'_l = x_l$ for every $l \neq i, j$.

An *inequality index* I is a non-trivial real-valued continuous function $I : D \rightarrow \mathbb{R}_+$ satisfying the following properties.

Pigou-Dalton Transfer Principle (TRP): $I(\mathbf{x}') < I(\mathbf{x})$ whenever $\mathbf{x}'$ is obtained from $\mathbf{x}$ by a progressive transfer.

Symmetry (SYM): $I(\mathbf{x}') = I(\mathbf{x})$ whenever $\mathbf{x}'$ is obtained from $\mathbf{x}$ after some permutation.

Normalization (NOR): $I(c, \ldots, c) = 0$ for all $c > 0$.

Replication Invariance (RIN): $I(.)$ does not change when a population is replicated $m \in \mathbb{N}$ times.[2]

The Pigou-Dalton Transfer Principle requires that a transfer from a richer country towards a poorer one (without altering their relative positions) should decrease inequality[3] .

[2] Whenever $\mathbf{x}' = (\mathbf{x}, \ldots, \mathbf{x})$ (m copies of $\mathbf{x}$ are repeated one after the other in the same vector), we say that $\mathbf{x}'$ is obtained after a $m-$time replication of $\mathbf{x} \in D$. RI requires that $I(\mathbf{x}') = I(\mathbf{x})$.

[3]

Symmetry establishes that an inequality index should not depend on an eventual reordering of the countries we are studying. Normalization requires inequality to be zero whenever all countries achieve the same level. Lastly, Replication Invariance allows populations of different sizes to be comparable. These four axioms have come to be accepted as the basic properties any inequality index should satisfy (see Chakravarty 2009).

We say that an inequality index I is *absolute* if for all $\mathbf{x} \in D, k \in \mathbb{R}$ one has that $I(\mathbf{x}+k) = I(\mathbf{x})$ whenever $\mathbf{x}+k \in D$. Alternatively, we say that an inequality index I is *relative* if for all $\mathbf{x} \in D$, $I(\lambda\mathbf{x}) = I(\mathbf{x})$ whenever $\lambda\mathbf{x} \in D$.

We now list some standard inequality indices that will be used throughout the paper.

$$\sigma^2(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{i=n}(x_i - \mu(\mathbf{x}))^2 \tag{1}$$

$$CV(\mathbf{x}) := \frac{\sigma(\mathbf{x})}{|\mu(\mathbf{x})|} \tag{2}$$

$$G_a(\mathbf{x}) := \frac{1}{2n^2}\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}|x_i - x_j| \tag{3}$$

$$G_r(\mathbf{x}) := \frac{1}{2\mu(\mathbf{x})n^2}\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}|x_i - x_j| \tag{4}$$

Equation (1) shows the formula of the variance. The equation shown in (2) corresponds to the coefficient of variation, which can be seen as a relative version of the standard deviation. In (3) and (4) we have the absolute and relative versions of the Gini coefficient respectively. Except for G_r, the values of these inequality indices are not bounded, they can take arbitrary big or small values (the larger they are, the higher the dispersion in $\mathbf{x}$). On the other hand,

Generally, the bounded indicators we discuss in this paper are not really transferable (e.g. we are not 'uneducating' highly educated individuals and transferring that education to less educated ones). Yet, one can compare the two scenarios (pre- and post-"transfers") as if they were from two different countries and still judge that the latter exhibits lower inequality than the former.

the relative Gini index is a bounded indicator taking values between 0 and 1. At one extreme, G_r takes a value of 1 when $x_i = 0$ for all individuals except for one. At the other extreme, when $x_i = c \forall i$ for some c, then G_r (and, by NOR, all inequality indices) go to 0.

Limitations of existing measures

As mentioned in the introduction the different inequality measures presented so far are affected by the boundary effects, and the attempts to solve the problem using relative inequality measures are fraught with inconsistency problems. In an attempt to solve the problem of strong mean-dependency in the study of lifespan variability, Monden and Smits (2009) introduced the so-called 'relative length of life inequality' index (RLI). The RLI is calculated standardizing the length of life inequality scores within one-year ranges of life expectancy, that is: it represents the deviation from average length of life inequality at a certain level of life expectancy in units of one standard deviation. While this immediately solves the problem of mean-dependency, it generates many other problems. Because of the way in which it is defined, the $RLI(\mathbf{x})$ of a given country depends not only on the corresponding $\mathbf{x}$ distribution, but also on the age at death distributions of other countries having the same life expectancy $\mu(\mathbf{x})$. This not only complicates its interpretation and compromises its comparability across countries and over time but also violates some of the basic axioms upon which all inequality indices are based, like the Pigou-Dalton transfers principle.

3. Benchmark inequality measures

Consider the following hypothetical distributions: $\mathbf{x} = (30, 40, 60, 70)$ and $\mathbf{y} = (80, 80, 100, 100)$. For the sake of the illustration, let us start thinking about them as income distributions in two hypothetical four-person societies. As is readily verified, traditional income inequality measures rank distribution $\mathbf{x}$ as being more unequal than distribution $\mathbf{y}$. To illustrate, one has that $\sigma(\mathbf{x}) = 18.26, \sigma(\mathbf{y}) = 11.55$, $CV(\mathbf{x}) = 0.37, CV(\mathbf{y}) = 0.13$, $G_a(\mathbf{x}) = 8.75, G_a(\mathbf{y}) = 5$ and $G_r(\mathbf{x}) = 0.18, G_r(\mathbf{y}) = 0.06$. Suppose now that $\mathbf{x}$ and $\mathbf{y}$ are no longer describing income distributions, but rather the percentage of vaccinated population in two hypothetical four-region societies. If this is the case, the underlying indicator we are working with is bounded between 0 and 100. While distribution $\mathbf{x}$ would represent an intermediate development stage (with an average $\mu(\mathbf{x}) = 50$) in which some regions have relatively high and others relatively low vaccination rates, distribution $\mathbf{y}$ would represent a more advanced development stage (with an average $\mu(\mathbf{y}) = 90$) in which most regions would have either achieved universal vaccination or would be very close to achieving it. At this point, one could argue that while it is true that distribution $\mathbf{y}$ seems to exhibit less dispersion than $\mathbf{x}$, it could hardly have been otherwise *given the average achievement levels in both distributions*. In other words: when the mean of a $0 - 100$ bounded indicator is set at 50, there is much more room for potential variability than what is feasible when that mean is set at 90 – a much higher value near the upper bound. In this framework, while an average of 50 can be obtained when pulling together such disparate values as 0 and 100, in the case of $\mu(\mathbf{y}) = 90$ the most disparate choice of that kind we can make is to take the pair of values 80 and 100. As is clear from this illustrative example, the mean value of a bounded distribution strongly conditions the range of values that traditional inequality measures can possibly take – an issue that severely difficults the comparison of inequality levels for distributions with differing means.

In the traditional unbounded case the comparison of distributions with differing means is easily achieved by resorting to relative inequality measures. Unfortunately, as shown by Lambert and Zheng (2011) relative measures fail to generate consistent estimates both for achievement and shortfall distributions. In this paper we suggest an alternative approach that circumvents the aforementioned problems. For any $\mathbf{x} \in [0, U]^n$ we compare observed inequality levels $I(\mathbf{x})$ with respect to the ones that would be observed under a hypothetical distribution *with mean* $\mu(\mathbf{x})$ that maximized $I(.)$. This way, we derive a 'relative-like' measure that compares observed inequality levels against a mean-dependent benchmark case. In order to operationalize this idea we need to define what is the hypothetical distribution maximizing any absolute inequality index satisfying TRP, SYM, NOR and RIN.

Proposition 1. Let $I : D^U \to \mathbb{R}_+$ be an absolute inequality index satisfying TRP, SYM, NOR and RIN. For a given $\mu \in [0, U]$, the distribution with mean μ maximizing inequality is a bimodal distribution where the population is split in two groups: the first one with a share s_1 attaining a value of 0 and the other one (with a share of $1 - s_1$) attaining U, in such a way that $s_1 \cdot 0 + (1 - s_1)U = \mu$. Such distribution will be denoted by $\boldsymbol{\xi}_\mu$.

Proof: See Appendix.

The distribution $\boldsymbol{\xi}_\mu$ represents a hypothetical scenario that would maximize inequality in the context of bounded variables. At one extreme, one portion of the population (with share s_1) attains the lowest achievement (0) and at the other extreme the remaining population attains the highest possible achievement (U). As the mean μ approaches the upper bound U, the share of the group with lowest achievement (s_1) gradually goes to zero. Even if it is a hypothetical distribution that is unlikely to be observed in the real world, $\boldsymbol{\xi}_\mu$ represents the benchmark case of extreme inequality against which we can compare our bounded distributions.

Definition 1: Let $I : D^U \to \mathbb{R}_+$ be an absolute measure of inequality satisfying TRP, SYM, NOR and RIN. For any $\mathbf{x} \in D^U$ we define the corresponding *benchmarked inequality index* as:

$$I^*(\mathbf{x}) := \frac{I(\mathbf{x})}{I(\boldsymbol{\xi}_{\mu(\mathbf{x})})} \tag{5}$$

with the convention that $I^*(\mathbf{0}) = I^*(\mathbf{U}) = 0$.

Because of the way in which it has been constructed, the new inequality measure takes values between 0 and 1: it compares the observed inequality level $I(\mathbf{x})$ with respect to the maximal value that, according to Proposition 1, $I(.)$ could possibly take. Importantly, it should be highlighted that this approach is only feasible in the bounded context (in the unbounded one it is not possible to construct the $\boldsymbol{\xi}_\mu$ distribution). In a way, the benchmark approach is reminiscent of the approach suggested by Monden and Smits (2009) when constructing the RLI index in their study of lifespan variation (see the last paragraph of section 2). In that paper, the authors also normalize the original index $I(\mathbf{x})$ in order to get rid of the distorting effects of the mean. However, while the normalization procedure they suggest can be said to be 'relative' (as it depends on other countries' distributions), our normalization can be said to be 'absolute' (as it is not affected by other countries' distributions). Using a fixed benchmark for all distributions having mean μ circumvents the technical problems referred to at the end of section 2. Indeed, the following results show that $I^*(.)$ satisfies several interesting properties.

Proposition 2.Whenever I satisfies the four basic inequality axioms (TRP, SYM, NOR and RIN), I^* satisfies them as well.

Proof: See Appendix.

As opposed to what happens with the RLI index the I^* suggested here is a proper inequality measure. Indeed, we suggest that I^* should be seen as the relative version of I adapted to the bounded framework (a point that is developed further after presenting equation (8)). We now show a couple of benchmarked inequality measures I^* defined on the basis of well-known families of absolute inequality indices $I(.)$.

1. The variance family

Consider the variance ($I = \sigma^2$). In that case it is easy to prove that $\sigma^2(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = \mu(\mathbf{x})(U - \mu(\mathbf{x}))$ (see appendix). Therefore, we can define

$$\left(\sigma^2\right)^*(\mathbf{x}) := \frac{\sigma^2(\mathbf{x})}{\mu(\mathbf{x})(U - \mu(\mathbf{x}))} \tag{6}$$

with the convention that $\left(\sigma^2\right)^*(\mathbf{0}) = \left(\sigma^2\right)^*(\mathbf{U}) = 0$. It is easy to prove that $\lim\limits_{\mu \to U} \left(\sigma^2\right)^* = 0, \lim\limits_{\mu \to 0} \left(\sigma^2\right)^* = 0$, so $\left(\sigma^2\right)^*$ is continuous for all $\mathbf{x} \in D^U$ (see appendix). The index $\left(\sigma^2\right)^*(\mathbf{x})$ simply measures the relative size of the variance $\sigma^2(\mathbf{x})$ with respect to the maximal value that such indicator could possible take for any distribution having mean $\mu(\mathbf{x})$. It is straightforward to check that $\left(\sigma^2\right)^*(\mathbf{x}) = \left(\sigma^2\right)^*(\mathbf{s})$ for all $\mathbf{x} \in D^U$, that is: $\left(\sigma^2\right)^*$ satisfies the so-called mirror property (see Erreygers 2009). This way, $\left(\sigma^2\right)^*$ is not affected by inconsistency problems.

2. The Gini family

Consider now the absolute Gini index ($I = G_a$). In that case it is easy to prove that $G_a(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = \frac{\mu(\mathbf{x})(U - \mu(\mathbf{x}))}{U}$ (see appendix). Therefore, we can define

$$G_a^*(\mathbf{x}) := \frac{G_a(\mathbf{x})}{\left(\frac{\mu(\mathbf{x})(U - \mu(\mathbf{x}))}{U}\right)} \tag{7}$$

with the convention that $G_a^*(\mathbf{0}) = G_a^*(\mathbf{U}) = 0$. It is easy to prove that $\lim\limits_{\mu \to U} G_a^* = 0, \lim\limits_{\mu \to 0} G_a^* = 0$, so G_a^* is continuous for all $\mathbf{x} \in D^U$ (see appendix). The index $G_a^*(\mathbf{x})$ measures the relative size of the absolute Gini index $G_a(\mathbf{x})$ with respect to the maximal value that such indicator could possible take for any distribution having mean $\mu(\mathbf{x})$. One can easily check that $G_a^*(\mathbf{x}) = G_a^*(\mathbf{s})$ for all $\mathbf{x} \in D^U$, so G_a^* also satisfies the mirror property and is unaffected by inconsistency problems. Lastly, one can easily show that

$$\lim_{U \to \infty} G_a^*(\mathbf{x}) = G_r(\mathbf{x}) \tag{8}$$

(see appendix). Equation (8) nicely links the bounded inequality measures introduced in this paper with the classical (i.e. unbounded) ones: as the upper bound is allowed to take arbitrarily large values, our benchmark inequality index converges towards the relative Gini index. This suggests that G_a^* is 'the right' counterpart of a relative inequality index in the context of bounded variables.

We conclude this section going back to the illustrative example with which we started it (where we compared inequality levels between distributions $\mathbf{x} = (30, 40, 60, 70)$ and $\mathbf{y} = (80, 80, 100, 100)$). We now have that $\sigma^*(\mathbf{x}) = 0.32 < \sigma^*(\mathbf{y}) = 0.33$ and $G_a^*(\mathbf{x}) = 0.35 < G_a^*(\mathbf{y}) = 0.56$. That is: contrary to what happens with traditional (unbounded) inequality measures, now it turns out that $\mathbf{y}$ has higher inequality levels than $\mathbf{x}$.

4. Is infant mortality converging worldwide?

The reduction of child mortality lies at the heart of the United Nations' Millennium Development Goals (MDGs) project. The fourth of these goals (MDG #4) committed the world nations to reduce by two thirds, between 1990 and 2015, the corresponding under-five mortality rates (U5MRs)[4] , a bounded indicator. While important progress has been made during the last decades to reduce the prevalence in child mortality (see Lozano et al 2011), such progress has been quite uneven. In this respect different studies suggest that global progress in demographic and public health indicators do not warrant convergence across countries (e.g. Neumayer 2004, McMichael et al. 2004, Becker et al. 2005; Moser et al. 2005, Dorius 2008). The extent to which world countries are converging in non-pecuniary dimensions of well-being (like child mortality levels) is often used as an acid test for the pur-

[4] The under-five mortality rate refers to the probability of dying before age 5 per 1000 newborns. While unorthodox, it is perfectly possible to describe our distributions using the under-five *survival* rates (for country i they are simply defined as $U5SR_i := 1000 - U5MR_i$, i.e. the probability of surviving to age 5 per 1000 newborns).

portedly beneficial impacts of economic globalization. In this section we investigate whether or not the world is converging in U5MR levels using our new benchmark inequality indices and comparing them with respect to traditional approaches[5] .

Using data from CME Info[6] we have U5MR estimates for 190 countries from 1950 to 2010. This allows a very precise description of the levels and trends in child mortality. Figure 1 shows different density functions of the global U5MR distribution in different moments in time: 1950, 1980 and 2010. As can be seen, major improvements have taken place around the world during the last 60 years. Back in the 1950s, the countries' U5MR distribution was very spread out and had two peaks. Six decades later, the distribution has shrunk considerably to the left, with a great majority of countries approaching the lower bound of 0. Inspecting the shape of the density functions shown in Figure 1 it seems clear that inequality must have decreased over time for absolute measures. Yet, the evolution of relative inequality measures can not be easily inferred by visual inspection.

[[[Figure 1_around_here]]]

In table 1 we show the mean levels and the values of different inequality indices (weighted by the corresponding population size) for the U5MR distributions (and occasionally their complement: U5SR) during the last decades. The global mean of U5MR has decreased from 198 in 1950 to 41.5 sixty years later (see column 1). As expected, the absolute inequality measures G_a and σ have decreased considerably – their values in 2010 are around one third of their original values in 1950 (see columns 2 and 3). According to these indices, the world is

[5] There are many ways of assessing global convergence in a given indicator (e.g. β-convergence and σ-convergence are very popular methods). As done in many other studies (e.g. Dorius 2008, Clark 2011), in this paper we will assess convergence by inspecting the evolution of global inequality over time.

[6] This is a database containing the latest child mortality estimates based on the research of the UN Inter-agency group for child mortality estimation (see www.childmortality.org).

converging in child mortality levels. What about relative inequality indices? In columns 4 to 7 we show the values of $G_r(U5MR), G_r(U5SR), CV(U5MR)$ and $CV(U5SR)$ over time As can be seen, the values of $G_r(U5MR)$ and $G_r(U5SR)$ go in opposite directions, and the same can be said about $CV(U5MR)$ and $CV(U5SR)$. When the mean of the $U5MR$ distribution approaches zero relative inequality increases, and the opposite occurs when, symmetrically, the mean of the $U5SR$ approaches 1000. Therefore, relative inequality measures are inconsistent and send contradictory messages as regards the international convergence/divergence in child mortality.

To circumvent the inconsistency problem experienced by relative measures we explore the values of our benchmark inequality indices, which can be seen as the equivalent of relative measures in the context of bounded variables. In columns 8 and 9 we show the trends of σ^* and G_a^* over time. Interestingly, these two indicators go in opposite directions: while σ^* suggests that countries are converging over time, the values of G_a^* suggest the opposite. Thus, the optimistic conclusions reached by looking at absolute measures are somewhat blurred when inspecting the evolution of benchmark inequality indices.

[[[Table 1_around_here]]]

5. Summary and concluding remarks

The use of traditional income inequality measures to study the variability of bounded variables poses several problems. On the one hand, the bounded nature of the variables generate a strong relationship between the mean of a distribution and its inequality levels (i.e. when the mean approaches either the lower or the upper bounds, inequality mechanically goes to zero). On the other hand, the lack of consistency between achievement and shortfall distributions precludes the use of classical measures of relative inequality. In this paper we propose new inequality indices that aim at capturing the intuitions of relative inequality but adapted to the context of bounded variables (the so-called benchmark inequality measures). The benchmark inequality indices proposed here simultaneously solve both problems: they consistently rank achievement and shortfall distributions and take into account the fact that when the mean of a bounded distribution approaches some of its bounds, the scope for potential variability is substantially reduced.

In an empirical application inspecting the global evolution of child mortality rates we observe that while absolute inequality indices suggest an overall inconditional convergence across countries, the results are not so optimistic when introducing our benchmark inequality measures. In some cases the new measures suggest international convergence but in others one finds that divergence is the observed trend. Interestingly these findings contrast with the trends in global income inequality, which is reported to increase when using absolute measures and to decrease with relative ones (see Niño-Zarazúa et al 2016).

We do not contend that the benchmark inequality measures proposed in this paper are superior to their absolute counterparts. Our proposal simply presents a complementary tool that is able to capture 'relative-like intuitions' that are currently missing in the case of bounded variables. Like in the unbounded case, the choice between absolute or relative measures is purely normative and none of them can claim superiority above the other.

6. Appendix

Proof of Proposition 1: This result is an immediate consequence of Theorem 1 in Seth and Yalonetzky (2016), which in turn is partially based on the work of Moyes (1987). According to that result, the absolute Lorenz curve of the distribution $\boldsymbol{\xi}_{\mu}$ will always be below the absolute Lorenz curve of any other distribution in $[0, U]$ with mean μ. This implies that all absolute inequality indices satisfying TRP, SYM, POP and NOR will deem $\boldsymbol{\xi}_\mu$ more unequal than any other distribution in $[0, U]$ with mean μ, as we wanted to demonstrate.

Proof of Proposition 2: We verify that when I satisfies TRP, SYM, NOR and RIN, then I^* satisfies them as well.

TRP: Assume $\mathbf{x}'$ is obtained from $\mathbf{x}$ by a progressive transfer. Since $\mu(\mathbf{x}) = \mu(\mathbf{x}')$, then $\boldsymbol{\xi}_{\mu(\mathbf{x})} = \boldsymbol{\xi}_{\mu(\mathbf{x}')}$, so $I^*(\mathbf{x}) = I(\mathbf{x})/I(\boldsymbol{\xi}_{\mu(\mathbf{x})}) \geq I(\mathbf{x}')/I(\boldsymbol{\xi}_{\mu(\mathbf{x}')}) = I^*(\mathbf{x}')$. Hence I^* satisfies TRP.

SYM: Assume $\mathbf{x}'$ is obtained from $\mathbf{x}$ after some permutation. Since $\mu(\mathbf{x}) = \mu(\mathbf{x}')$, then $\boldsymbol{\xi}_{\mu(\mathbf{x})} = \boldsymbol{\xi}_{\mu(\mathbf{x}')}$, so $I^*(\mathbf{x}) = I(\mathbf{x})/I(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = I(\mathbf{x}')/I(\boldsymbol{\xi}_{\mu(\mathbf{x}')}) = I^*(\mathbf{x}')$. Hence I^* satisfies SYM.

NOR: Take any $c > 0$ and consider the constant distribution $\mathbf{c} = (c, \ldots, c)$. Then $I^*(\mathbf{c}) = I(\mathbf{c})/I(\boldsymbol{\xi}_c) = 0$, so I^* satisfies NOR.

RIN: Assume $\mathbf{x}'$ is obtained after replicating $\mathbf{x}$ $m \in \mathbb{N}$ times. Since $\mu(\mathbf{x}) = \mu(\mathbf{x}')$, then $\boldsymbol{\xi}_{\mu(\mathbf{x})} = \boldsymbol{\xi}_{\mu(\mathbf{x}')}$, so $I^*(\mathbf{x}) = I(\mathbf{x})/I(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = I(\mathbf{x}')/I(\boldsymbol{\xi}_{\mu(\mathbf{x}')}) = I^*(\mathbf{x}')$. Hence I^* satisfies RIN.

Results for the benchmark inequality measure σ^*

Statement 1: $\sigma^2(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = \mu(\mathbf{x})(U - \mu(\mathbf{x}))$.

Proof: By definition, one has that (i) $\sigma^2(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = s_1(0 - \mu(\mathbf{x}))^2 + s_2(U - \mu(\mathbf{x}))^2$ and (ii) $s_1 0 + s_2 U = \mu(\mathbf{x})$. After trivial algebraic manipulations one gets the desired result.

Statement 2: $(\sigma^2)^*$ is continuous for all $\mathbf{x} \in D^U$.

Proof: It is obvious that $(\sigma^2)^*$ is continuous for all $\mathbf{x} \in D^U \backslash \{\mathbf{0}, \mathbf{U}\}$, so we only need to check continuity at $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{U}$. By definition, $\lim\limits_{\mu \to U} (\sigma^2)^* = \lim\limits_{\mu \to U} \frac{\sigma^2(\mathbf{x})}{\mu(\mathbf{x})(U - \mu(\mathbf{x}))} = \frac{0}{0}$. To solve this indeterminacy we apply L'Hôspital's rule to the last expression:

$$\lim_{\mu \to U} \frac{\sigma^2(\mathbf{x})}{\mu(\mathbf{x})(U - \mu(\mathbf{x}))} = \lim_{\mu \to U} \frac{\partial(\sigma^2(\mathbf{x}))/\partial\mu}{\partial(\mu(\mathbf{x})(U - \mu(\mathbf{x})))/\partial\mu} = \lim_{\mu \to U} \frac{-\sum_i 2(x_i - \mu)}{n(U - 2\mu)} \tag{9}$$

Since $-\sum_i 2(x_i - \mu) = -2(\sum_i x_i - \sum_i \mu) = -2(n\mu - n\mu) = 0$, the limit shown in equation (9) goes to 0, so $(\sigma^2)^*$ is continuous at $\mathbf{x} = \mathbf{U}$. Using exactly the same arguments we conclude that $(\sigma^2)^*$ is also continuous at $\mathbf{x} = \mathbf{0}$.

Results for the benchmark inequality measure G_a^*

Statement 3: $G_a(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = \mu(\mathbf{x})(U - \mu(\mathbf{x}))/U$.

Proof: Define $n_1 := s_1 n$ and $n_2 := s_2 n$. Clearly $n_1 + n_2 = n$. By definition, one has that

$$G_a(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = \frac{2n_1 n_2 |U - 0|}{2n^2} = s_1 s_2 U \tag{10}$$

In addition, since $s_1 0 + s_2 U = \mu(\mathbf{x})$, we can rewrite (10) as

$$G_a(\boldsymbol{\xi}_{\mu(\mathbf{x})}) = (1 - s_2) s_2 U = \left(1 - \frac{\mu(\mathbf{x})}{U}\right) \frac{\mu(\mathbf{x})}{U} U = \frac{\mu(\mathbf{x})(U - \mu(\mathbf{x}))}{U}. \tag{11}$$

Statement 4: G_a^* is continuous for all $\mathbf{x} \in D^U$.

Proof: It is obvious that G_a^* is continuous for all $\mathbf{x} \in D^U \backslash \{\mathbf{0}, \mathbf{U}\}$, so we only need to check continuity at $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{U}$. By definition, $\lim_{\mu \to U} G_a^* = \lim_{\mu \to U} \frac{G_a(\mathbf{x})}{\mu(\mathbf{x})(U - \mu(\mathbf{x}))/U} = \frac{0}{0}$. To solve this indeterminacy we apply L'Hôspital's rule to the last expression:

$$\lim_{\mu \to U} \frac{G_a(\mathbf{x})}{\mu(\mathbf{x})(U - \mu(\mathbf{x}))/U} = \lim_{\mu \to U} \frac{\partial G_a(\mathbf{x})/\partial\mu}{\partial(\mu(\mathbf{x})(U - \mu(\mathbf{x}))/U)/\partial\mu} = \lim_{\mu \to U} \frac{0}{(U - 2\mu(\mathbf{x}))/U} = 0 \tag{12}$$

so G_a^* is continuous at $\mathbf{x} = \mathbf{U}$. Using exactly the same arguments we conclude that G_a^* is also continuous at $\mathbf{x} = \mathbf{0}$.

Statement 5: $\lim_{U \to \infty} G_a^*(\mathbf{x}) = G_r(\mathbf{x})$.

Proof: By definition one has that

$$\lim_{U\to\infty} G_a^*(\mathbf{x}) = \lim_{U\to\infty} \frac{UG_a(\mathbf{x})}{\mu(\mathbf{x})(U-\mu(\mathbf{x}))} = \frac{\infty}{\infty} \tag{13}$$

Applying L'Hôspital's rule to (13) we obtain the desired expression

$$\lim_{U\to\infty} G_a^*(\mathbf{x}) = \lim_{U\to\infty} \frac{\partial(UG_a(\mathbf{x}))/\partial U}{\partial(\mu(\mathbf{x})(U-\mu(\mathbf{x})))/\partial U} = \lim_{U\to\infty} \frac{G_a(\mathbf{x})}{\mu(\mathbf{x})} = G_r(\mathbf{x}) \tag{14}$$

Acknowledgements: The research leading to these results has received funding from the European Research Council (ERC-2014-StG-637768, EQUALIZE project), the CERCA Programme from Generalitat de Catalunya and from the Spanish Ministry of Economy and Competitiveness "Ramón y Cajal" Research Grant Program (RYC-2013-14196).

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Year	(1) μ	(2) G_a	(3) σ	(4) $G_r(U5MR)$	(5) $G_r(U5SR)$	(6) $CV(U5MR)$	(7) $CV(U5SR)$	(8) σ^*	(9) G_a^*
1950	197.99	57.66	102.20	0.29	0.07	0.52	0.13	0.26	0.36
1960	166.29	50.61	92.03	0.30	0.06	0.55	0.11	0.25	0.37
1970	114.93	42.36	75.78	0.37	0.05	0.66	0.09	0.24	0.42
1980	88.98	34.70	63.18	0.39	0.04	0.71	0.07	0.22	0.43
1990	70.35	29.33	55.48	0.42	0.03	0.79	0.06	0.22	0.45
2000	54.42	23.95	46.35	0.44	0.03	0.85	0.05	0.20	0.47
2010	41.50	18.94	37.28	0.46	0.02	0.90	0.04	0.19	0.48

Table 1. Mean and different inequality indices of the global IMR distribution: 1950-2010. Source: Author own elaboration based on CEM Info data.

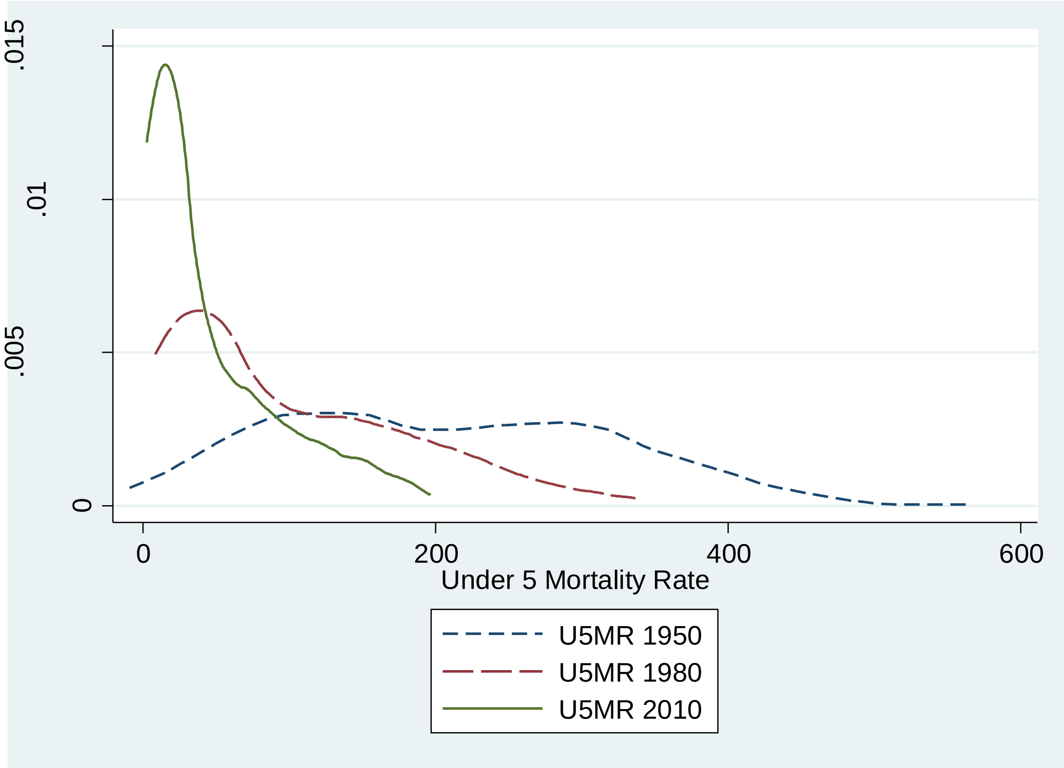

Figure 1. Density functions of global under-five mortality rate distributions in 1950, 1980 and 2010. Source: Author own elaboration based on CEM Info data.